SYDNEY GRAMMAR SCHOOL

CANDIDATE NUMBER

2022 Trial Examination

Form VI Mathematics Advanced

Wednesday 10th August 2022
8:40am

General Instructions

- Reading time — 10 minutes
- Working time — 3 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.
- Remove the central staple: you should have this cover booklet with Section I, 4 booklets for Section II, and the multiple-choice answer sheet.

Total Marks: 100

Section I (10 marks) Questions 1 – 10

- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (90 marks) Questions 11 – 36

- Relevant mathematical reasoning and calculations are required.
- Answer the questions in this paper in the spaces provided.
- This section is divided into four parts. Extra writing paper is provided at the end of each part.

Collection

- Write your candidate number on this page, on the start of each separate part, and on the multiple choice sheet.

Checklist

- Reference sheet
- Multiple-choice answer sheet
- Candidature: 96 pupils

Writer: BLR

Section I

Questions in this section are multiple-choice.
Record the single best answer for each question on the provided answer sheet.

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1. Which of the following is the angle of inclination of the line $y = \frac{9}{8}x - 5$?

   (A) 42°

   (B) 48°

   (C) 132°

   (D) 138°

2. Which of the following correctly describes the gradient and concavity of the function $f(x) = 2x^2 + 3x - 1$ at the origin?

   (A) The gradient is positive and the function is concave up

   (B) The gradient is positive and the function is concave down

   (C) The gradient is negative and the function is concave up

   (D) The gradient is negative and the function is concave down

3. Which of the following is an odd function?

   (A) $y = x^2 + x$

   (B) $y = x^2 + 1$

   (C) $y = x^3 + x$

   (D) $y = x^3 + 1$

4. The n^{th} term of a series is given by $T_n = 2n^2 - 4$ for $n \geq 1$. What is the sum of the first three terms?

   (A) -6

   (B) 12

   (C) 14

   (D) 16

5. If $f(x) = x^2 - 2x$ and $g(x) = 2x + 1$, which of the following is a correct expression for $f(g(x))$?

   (A) $4x^2$

   (B) $4x^2 - 1$

   (C) $4x^2 - 4x - 1$

   (D) $4x^2 + 1$

6. A ghost gum tree was planted in 2010. The height of the tree H, in metres, is a function of the tree's age, in years, after planting. Which of the following is the correct interpretation of $H'(4) = 3{\cdot}2$?

   (A) Between 2010 and 2014, the height increased at a rate of 3·2 m per year

   (B) Between 2010 and 2014, the height increased by 3·2 m

   (C) At the start of 2014, the height was increasing at a rate of 3·2 m per year

   (D) At the start of 2014, the height was 3·2 m

7. How many solutions does the equation $\sin 3x = 0$ have for $0 \leq x \leq \pi$?

   (A) 0

   (B) 2

   (C) 4

   (D) 5

8. Which of the following is the correct simplification of $\dfrac{\operatorname{cosec}\theta \sec\theta}{1 + \tan^2\theta}$?

   (A) $-\tan\theta$

   (B) $-\cot\theta$

   (C) $\tan\theta$

   (D) $\cot\theta$

9. A differentiable function has a local minimum turning point at $(6, -2)$ and a local maximum turning point at $(-5, 9)$. Which of the following is NOT NECESSARILY true?

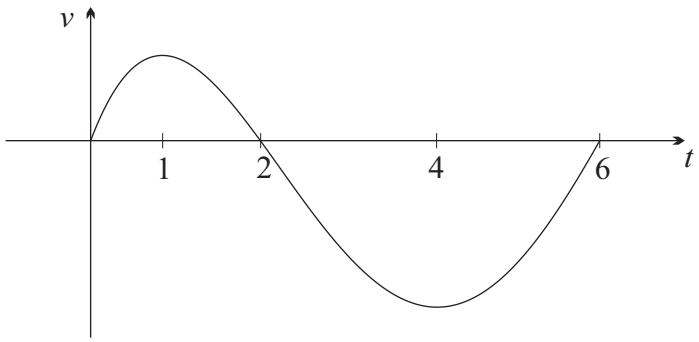
   (A) $f'(-5) = 0$ and $f'(6) = 0$

   (B) The function has a point of inflection between $x = -5$ and $x = 6$

   (C) The range of the function is $[-2, 9]$

   (D) The function has an x-intercept and a y-intercept

10. A ball moves in a straight line along a horizontal track. The graph below shows the ball's velocity v at time t.



For what values of t is the speed of the particle decreasing?

(A) $0 < t < 1$ and $2 < t < 4$

(B) $1 < t < 2$ and $4 < t < 6$

(C) $1 < t < 4$

(D) $2 < t < 6$

End of Section I

The paper continues in the next section

QUESTION ELEVEN (1 mark) **Marks**

Find the domain of the function $f(x) = \sqrt{3x - 2}$. 1

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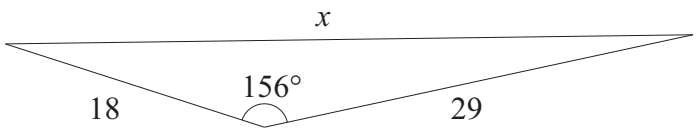
QUESTION TWELVE (2 marks) **Marks**

Find the value of x, correct to the nearest whole number. 2



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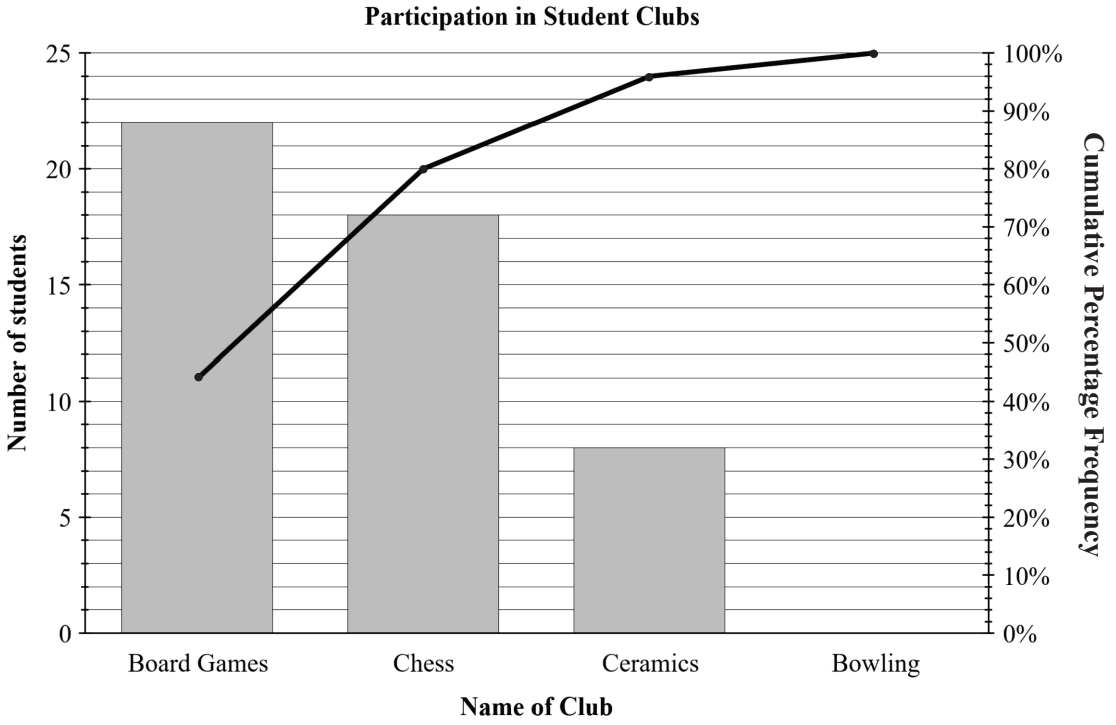
QUESTION THIRTEEN (2 marks) **Marks**

A group of 50 students are surveyed about their membership in student clubs. The diagram below shows an incomplete Pareto chart summarising the number of students who participate in different school clubs. Each student is a member of only one club.



(a) Complete the Pareto chart on the diagram above by adding the final column to represent membership of the bowling club. 1

(b) Which club has 16% of the students? 1

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QUESTION FOURTEEN (2 marks) **Marks**

Calculate the limiting sum of the geometric series $100 + 20 + 4 + \cdots$. 2

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QUESTION FIFTEEN (3 marks) **Marks**

Consider the function $f(x) = \ln(x-3)$.

(a) Find $f'(x)$. 1

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(b) Hence, find the equation of the tangent to the graph of $y = f(x)$ where $x = 4$. 2

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QUESTION SIXTEEN (2 marks) **Marks**

The probability distribution of a discrete random variable X is given in the table below. 2

x	4	9	14	a
$P(X = x)$	0·2	0·3	0·4	0·1

Find the value of a given that $E(X) = 10{\cdot}6$.

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QUESTION SEVENTEEN (2 marks) **Marks**

Solve $3^{3x-1} = 27$. 2

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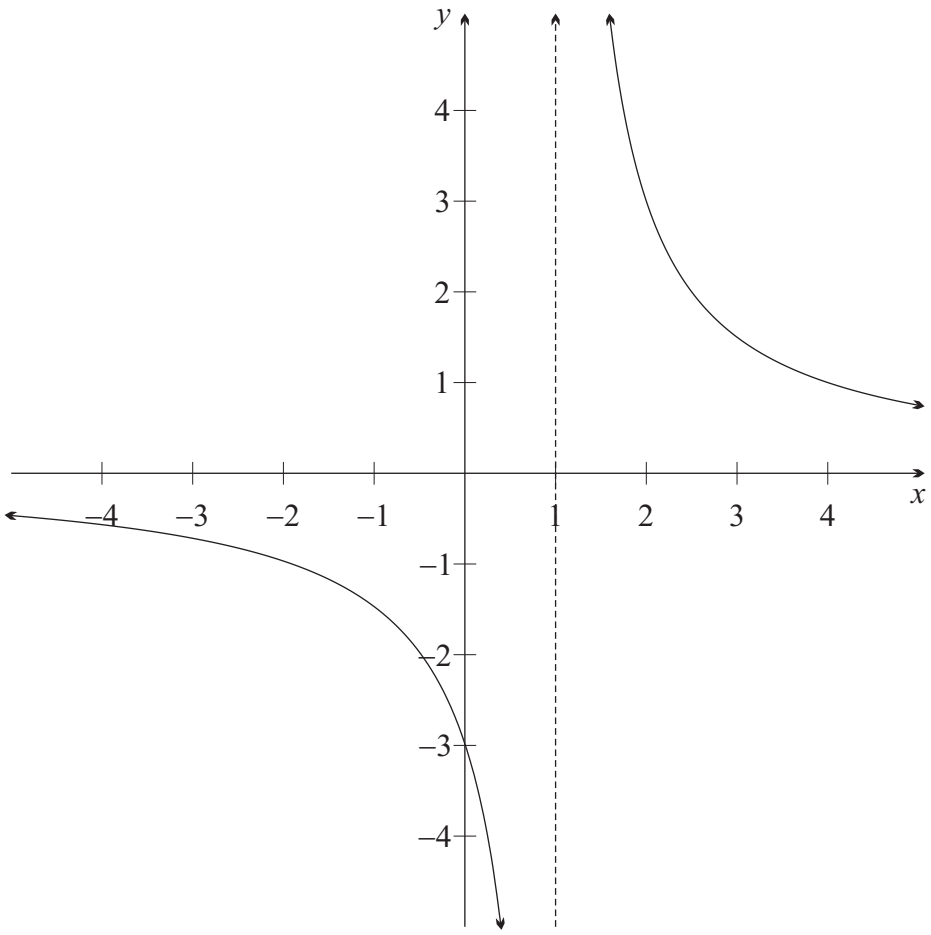
QUESTION EIGHTEEN (3 marks) **Marks**

The function $y = \dfrac{3}{x-1}$ has been graphed on the coordinate plane below.



(a) Sketch the graph of $y = |x| - 2$ on the coordinate plane above, clearly showing any intercepts with the axes. 2

(b) Hence, determine the number of solutions of the equation $|x| - \dfrac{3}{x-1} = 2$. 1

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QUESTION NINETEEN (5 marks) **Marks**

A company wanted to investigate the strength of the relationship between the amount of money spent on advertising each week and the number of visitors the company's website receives each week. Data was collected over a 15 week period.

(a) Pearson's correlation coefficient for the data set is $r = 0{\cdot}9$. Describe the strength and direction of this linear correlation. [1]

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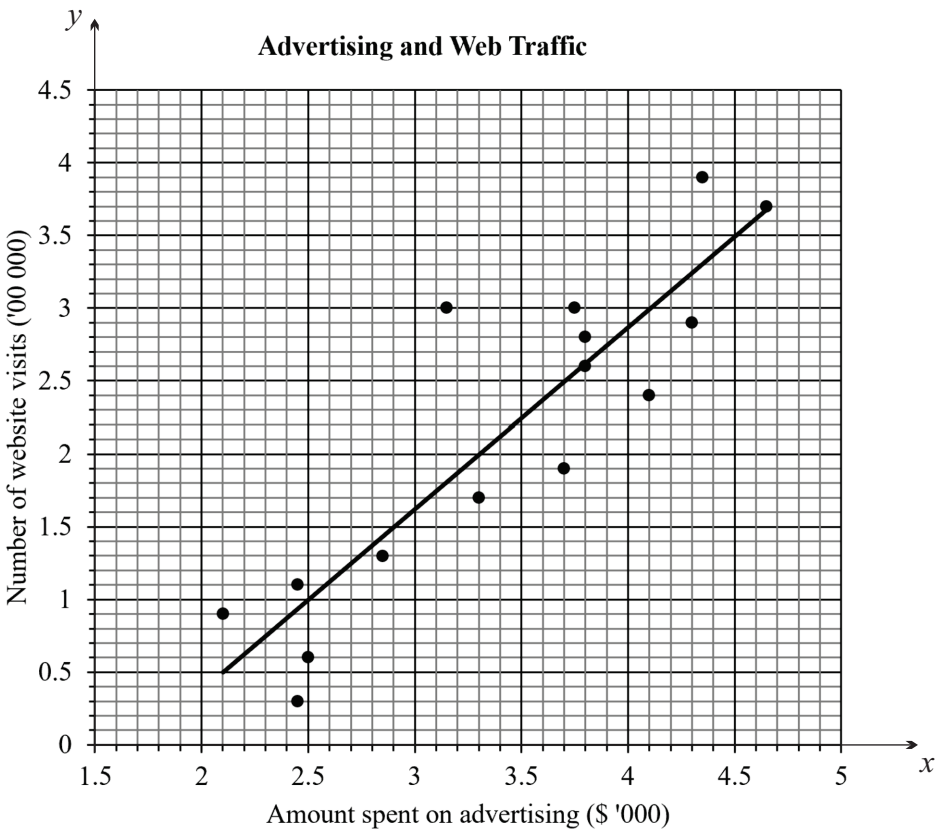
(b) The graph below shows a scatterplot and the regression line for the data, where x is the amount of money spent on advertising in thousands of dollars and y is the website traffic in hundreds of thousands of visits. [2]



Calculate the equation of the regression line.

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(c) Use the equation of your regression line from part (b) to predict the number of visitors the website will receive for a week in which \$3000 was spent on advertising. 1

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(d) Explain why the model is not useful for predicting the number of visitors for weeks in which \$1500 was spent on advertising. 1

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QUESTION TWENTY (2 marks) **Marks**

Find these indefinite integrals.

(a) $\int \left(6x^2 + 5\right)\, dx$ 1

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(b) $\int e^{4x}\, dx$ 1

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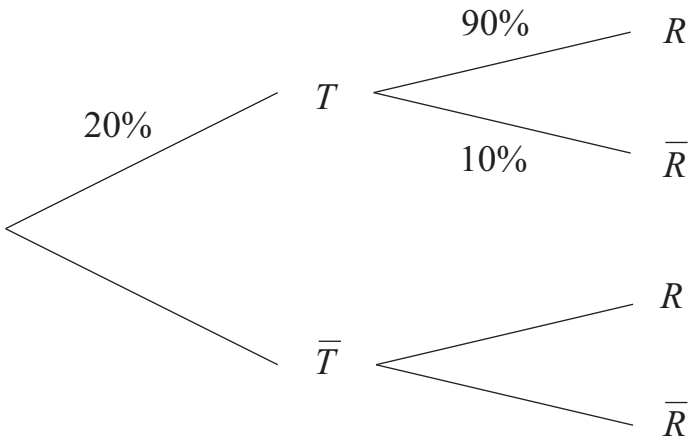
Extra writing space (Use this space only for questions in Part A)

If you use this space, clearly indicate which question you are answering.

QUESTION TWENTY-ONE (3 marks) **Marks**

Briana is on a hiking trip in the outback. She is within 50 km of a town 20% of the time. When she is within 50 km of a town, she has mobile reception 90% of the time. Otherwise, she has mobile reception 40% of the time.

(a) Let T be the event that Briana is within 50 km of a town and R be the event that she has mobile reception. Complete the probability tree diagram below by writing the missing probabilities on the branches. 1



(b) Calculate the probability that she has mobile reception at any given time. 2

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QUESTION TWENTY-TWO (2 marks) **Marks**

Expand and simplify $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$. 2

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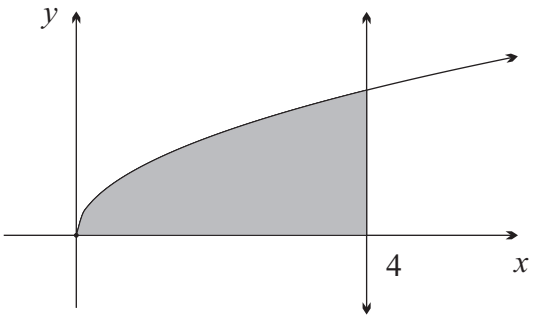
QUESTION TWENTY-THREE (2 marks) **Marks**

The diagram below shows the graph of the function $y = \sqrt{x}$. The shaded region is bounded by the curve, the x-axis, and the line $x = 4$. Calculate the area of the shaded region. [2]



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QUESTION TWENTY-FOUR (3 marks) **Marks**

The sum of the first n terms of an arithmetic series is S_n. For a particular arithmetic series, $S_8 = 196$ and $S_9 = 279$.

(a) Calculate the value of the 9^{th} term. 1

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(b) Hence, find the values of the first term a and the common difference d. 2

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QUESTION TWENTY-FIVE (1 mark) **Marks**

If $\int_{-10}^{4} f(x)\,dx = 3$ and $\int_{4}^{6} f(x)\,dx = 5$, find the value of $\int_{-10}^{6} 2f(x)\,dx$. 1

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QUESTION TWENTY-SIX (3 marks) **Marks**

The Richter scale is used to measure the magnitude of earthquakes. The magnitude R of an earthquake is proportional to the logarithm of the ground displacement d caused by the earthquake in millimetres. This relationship can be described by the equation:

$$R = \log_{10}(1000d)$$

(a) In July 2022, an earthquake in the Easter Island region caused a ground displacement of 6310 mm. What was its magnitude on the Richter scale? Give your answer correct to one decimal place. [1]

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(b) How many times larger is the ground displacement of an earthquake with a magnitude of 3·2 compared to an earthquake with a magnitude of 1·6 ? Give your answer correct to the nearest whole number. [2]

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QUESTION TWENTY-SEVEN (1 mark) **Marks**

The graph of the function $y = \dfrac{x^2 + 3x}{x^2 + 3}$ has a horizontal asymptote. Find the equation of this asymptote, clearly showing your working. [1]

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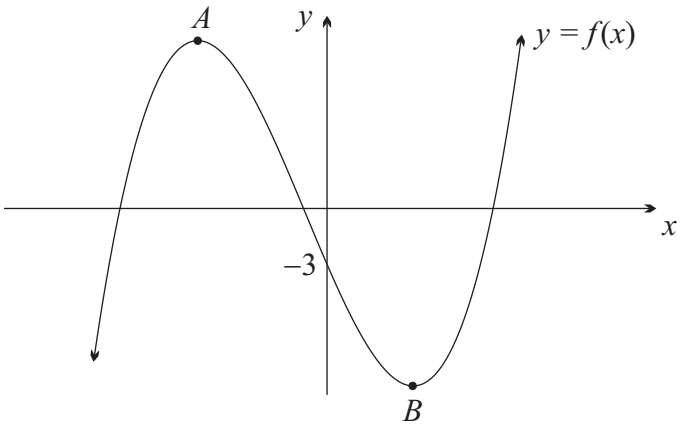
QUESTION TWENTY-EIGHT (6 marks) **Marks**

The diagram below shows the graph of a function $f(x)$ with a y-intercept of -3, a maximum turning point at A, and a minimum turning point at B. The derivative of the function is $f'(x) = 3x^2 + 2x - 8$.



(a) Find the equation of $f(x)$. [2]

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(b) Find the coordinates of A. 3

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(c) The graph of $y = g(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis, shifting 5 units up, and then dilating horizontally by a factor of $\frac{1}{2}$. 2
Find the coordinates of the point A', the image of the point A, after these successive transformations have been applied.

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Extra writing space (Use this space only for questions in Part B)

If you use this space, clearly indicate which question you are answering.

QUESTION TWENTY-NINE (2 marks) **Marks**

Consider the exact values $\sin\frac{\pi}{8} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$ and $\cos\frac{\pi}{8} = \dfrac{\left(\sqrt{2}-1\right)\sqrt{2+\sqrt{2}}}{2}$. 2

Show that $-\sin\frac{\pi}{8} - \cos\frac{\pi}{8} = -\dfrac{\sqrt{4+2\sqrt{2}}}{2}$.

QUESTION THIRTY (7 marks) **Marks**

Consider the function $f(x) = x^3 - 4x^2 + 4x$.

(a) Find the coordinates of the stationary points and determine their nature. 4

(b) Sketch the graph of $y = f(x)$, clearly showing any intercepts and stationary points. [2]

(c) Hence, determine the values of k for which the equation $x^3 - 4x^2 + 4x + k = 0$ has exactly two solutions. [1]

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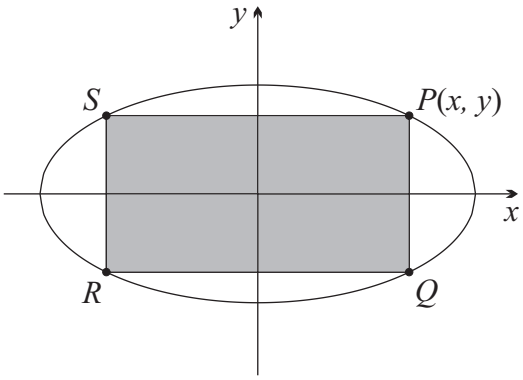
QUESTION THIRTY-ONE (7 marks) **Marks**

Rectangle $PQRS$ has been inscribed inside an ellipse with equation $x^2 + 4y^2 = 36$, as shown in the diagram below. The point P has coordinates (x, y).



(a) Show that the area of the rectangle is $A = 2x\sqrt{36 - x^2}$. 2

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(b) Show that $\dfrac{dA}{dx} = \dfrac{72 - 4x^2}{\sqrt{36 - x^2}}$. 2

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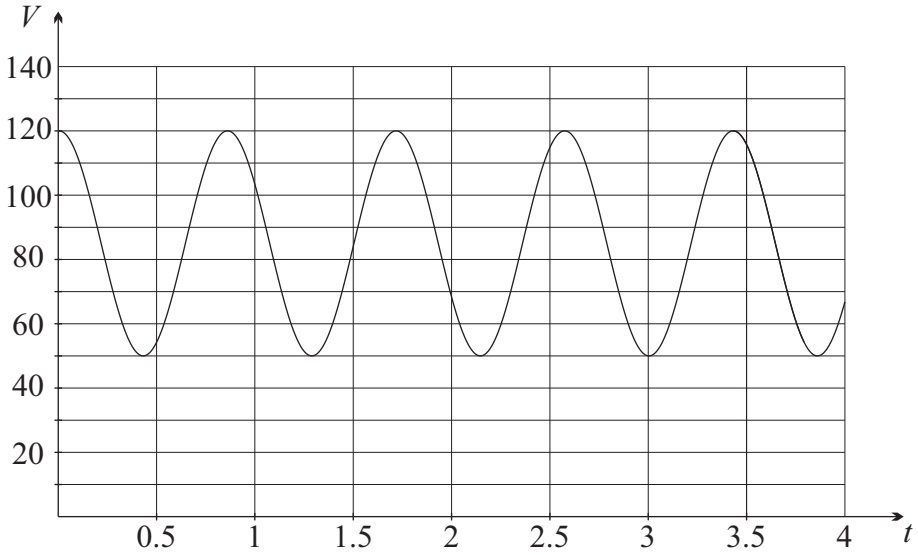
(c) Hence, find the maximum possible area of the rectangle. 3

QUESTION THIRTY-TWO (8 marks) **Marks**

The human heart pumps blood through the body by repeatedly contracting, which pushes blood out of the heart, and then relaxing, which lets new blood flow into the heart. The average human heart contains 50 mL of blood when contracted and 120 mL when relaxed. The total volume V mL of blood in the heart at time t seconds can be modelled by the function

$$V = 35\cos\left(\frac{7\pi}{3}t\right) + c$$

where c is a constant. The diagram below shows the graph of $V(t)$ for $0 \leq t \leq 4$.



(a) Write down the value of c. 1

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(b) Calculate the period of V. 1

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(c) Calculate the rate of change of volume when $t = \frac{1}{2}$. Give your answer correct to the nearest millilitre per second. 2

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(d) What is the total volume of blood which is pumped out of the heart between $t = \frac{2}{7}$ and $t = \frac{3}{7}$? Give your answer correct to the nearest millilitre. 2

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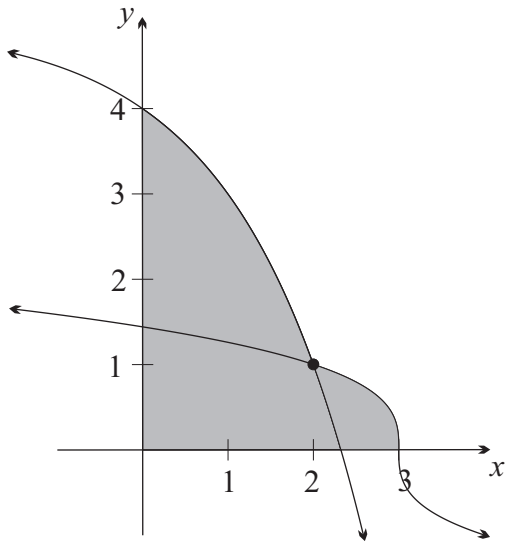
(e) By considering the volume of blood in the heart between $t = 0$ and $t = \frac{3}{7}$, or otherwise, determine what fraction of the time the heart contains at least 102·5 mL of blood. 2

QUESTION THIRTY-THREE (4 marks) **Marks**

The diagram below shows the graphs of $y = 5 - 2^x$ and $y = \sqrt[3]{3 - x}$. The graphs intersect at the point $(2, 1)$. Calculate the exact area of the shaded region. [4]



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QUESTION THIRTY-FOUR (3 marks) **Marks**

Show that the function $y = kxe^{kx}$ has exactly one stationary point and its y-value is independent of k, where k is a constant. [3]

QUESTION THIRTY-FIVE (3 marks) **Marks**

Arthur's bag contains 4 green marbles and 2 red marbles. Beatrice's bag contains 3 green marbles and 3 red marbles. One of their bags is randomly selected, and then two marbles are removed in succession without replacement. 3

What is the probability that Arthur's bag was selected, given that the two marbles were the same colour?

QUESTION THIRTY-SIX (10 marks) **Marks**

A table of present value interest factors for an annuity of \$1 is shown.

Table of present value interest factors

	Rate per time period			
Number of periods	0·05%	0·1%	0·15%	0·2%
120	116·44	113·03	109·74	106·59
180	172·10	164·65	157·64	151·04
240	226·11	213·28	201·42	190·46
300	278·52	259·07	241·44	225·43
360	329·38	302·20	278·01	256·45

On 1st April 2020, Ferdinand takes out a 25 year loan with reducible interest charged at a rate of 1·2% per annum, compounded monthly.
The loan is to be repaid in equal monthly repayments of \$2370 at the end of each month.

(a) Use the present value table to show that Ferdinand borrowed \$614 000, correct to the nearest thousand dollars. 1

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(b) Hence, calculate the total amount of interest Ferdinand will pay over the 25 year term of the loan. 2

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(c) Let A_n be the amount left to repay at the end of n months. 1

Show that $A_2 = 614\,000 \times 1{\cdot}001^2 - 2370\,(1 + 1{\cdot}001)$.

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(d) Show that $A_n = 614\,000 \times 1{\cdot}001^n - 2\,370\,000\,(1{\cdot}001^n - 1)$. 2

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SYDNEY GRAMMAR SCHOOL

SOLUTIONS

CANDIDATE NUMBER

2022 Trial Examination

Form VI Mathematics Advanced

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8:40am

General Instructions

- Reading time — 10 minutes
- Working time — 3 hours
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- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.
- Remove the central staple: you should have this cover booklet with Section I, 4 booklets for Section II, and the multiple-choice answer sheet.

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- This section is multiple-choice. Each question is worth 1 mark.
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Section II (90 marks) Questions 11 – 36

- Relevant mathematical reasoning and calculations are required.
- Answer the questions in this paper in the spaces provided.
- This section is divided into four parts. Extra writing paper is provided at the end of each part.

Collection

- Write your candidate number on this page, on the start of each separate part, and on the multiple choice sheet.

Checklist

- Reference sheet
- Multiple-choice answer sheet
- Candidature: 96 pupils

Writer: BLR

	Marks
Multiple Choice	/10
Part A	/24
Part B	/22
Part C	/24
Part D	/20
TOTAL	

Section I

Questions in this section are multiple-choice.
Record the single best answer for each question on the provided answer sheet.

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1. Which of the following is the angle of inclination of the line $y = \frac{9}{8}x - 5$?

   (A) $42°$

   (B) $48°$

   (C) $132°$

   (D) $138°$

$\tan\theta = \frac{9}{8}$

$\theta \doteqdot 48°$

2. Which of the following correctly describes the gradient and concavity of the function $f(x) = 2x^2 + 3x - 1$ at the origin?

   (A) The gradient is positive and the function is concave up

   (B) The gradient is positive and the function is concave down

   (C) The gradient is negative and the function is concave up

   (D) The gradient is negative and the function is concave down

$f'(x) = 4x + 3$

$f'(0) = 3$

$f''(x) = 4$

3. Which of the following is an odd function?

   (A) $y = x^2 + x$

   (B) $y = x^2 + 1$

   (C) $y = x^3 + x$

   (D) $y = x^3 + 1$

4. The n^{th} term of a series is given by $T_n = 2n^2 - 4$ for $n \geq 1$. What is the sum of the first three terms?

   (A) -6

   (B) 12

   (C) 14

   (D) 16

$T_1 = 2(1)^2 - 4$
$= -2$

$T_2 = 2(2)^2 - 4$
$= 4$

$T_3 = 2(3)^2 - 4$
$= 14$

$T_1 + T_2 + T_3$
$= -2 + 4 + 14$
$= 16$

5. If $f(x) = x^2 - 2x$ and $g(x) = 2x + 1$, which of the following is a correct expression for $f(g(x))$?

$f(2x+1) = (2x+1)^2 - 2(2x+1)$

$= 4x^2 + 4x + 1 - 4x - 2$

$= 4x^2 - 1$

(A) $4x^2$

(B) $4x^2 - 1$

(C) $4x^2 - 4x - 1$

(D) $4x^2 + 1$

6. A ghost gum tree was planted in 2010. The height of the tree H, in metres, is a function of the tree's age, in years, after planting. Which of the following is the correct interpretation of $H'(4) = 3{\cdot}2$?

(A) Between 2010 and 2014, the height increased at a rate of 3·2 m per year

(B) Between 2010 and 2014, the height increased by 3·2 m

(C) At the start of 2014, the height was increasing at a rate of 3·2 m per year

(D) At the start of 2014, the height was 3·2 m

7. How many solutions does the equation $\sin 3x = 0$ have for $0 \leq x \leq \pi$?

$0 \leq 3x \leq 3\pi$

(A) 0

(B) 2

(C) 4

(D) 5

8. Which of the following is the correct simplification of $\dfrac{\operatorname{cosec}\theta \sec\theta}{1 + \tan^2\theta}$?

$= \dfrac{\operatorname{cosec}\theta \sec\theta}{\sec^2\theta}$

$= \dfrac{\operatorname{cosec}\theta}{\sec\theta} = \dfrac{\frac{1}{\sin\theta}}{\frac{1}{\cos\theta}} = \dfrac{\cos\theta}{\sin\theta}$

(A) $-\tan\theta$

(B) $-\cot\theta$

(C) $\tan\theta$

(D) $\cot\theta$

9. A differentiable function has a local minimum turning point at $(6, -2)$ and a local maximum turning point at $(-5, 9)$. Which of the following is NOT NECESSARILY true?

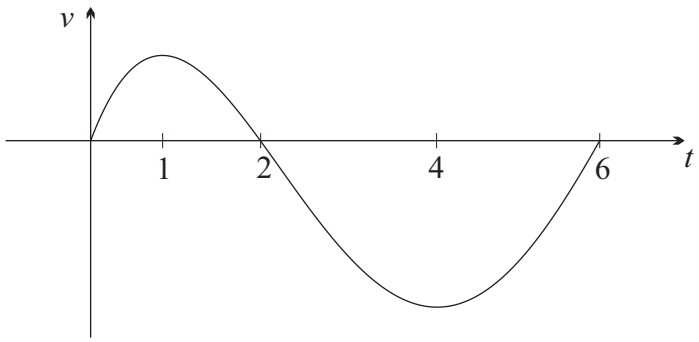
(A) $f'(-5) = 0$ and $f'(6) = 0$

(B) The function has a point of inflection between $x = -5$ and $x = 6$

(C) The range of the function is $[-2, 9]$

(D) The function has an x-intercept and a y-intercept

10. A ball moves in a straight line along a horizontal track. The graph below shows the ball's velocity v at time t.



For what values of t is the speed of the particle decreasing?

(A) $0 < t < 1$ and $2 < t < 4$

(B) $1 < t < 2$ and $4 < t < 6$

(C) $1 < t < 4$

(D) $2 < t < 6$

End of Section I

The paper continues in the next section

QUESTION ELEVEN (1 mark) **Marks**

Find the domain of the function $f(x) = \sqrt{3x-2}$. [1]

$3x - 2 \geq 0$

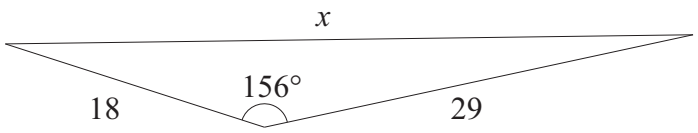
$3x \geq 2$

$x \geq \frac{2}{3}$ ✓

Common error: $x > \frac{2}{3}$

QUESTION TWELVE (2 marks) **Marks**

Find the value of x, correct to the nearest whole number. [2]



$x^2 = 18^2 + 29^2 - 2 \times 18 \times 29 \times \cos 156°$ ✓

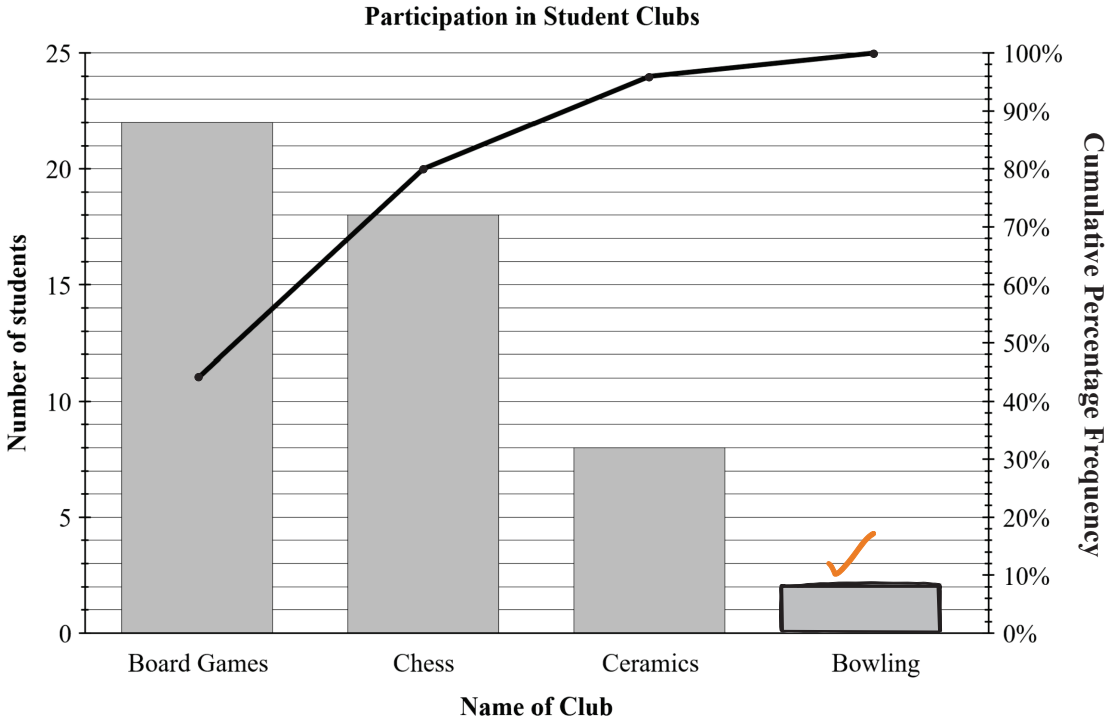
$x = \sqrt{1165 - 1044 \cos 156°}$

$x = 46.0297...$

$x \doteq 46$ ✓ with correct supporting working

QUESTION THIRTEEN (2 marks) **Marks**

A group of 50 students are surveyed about their membership in student clubs. The diagram below shows an incomplete Pareto chart summarising the number of students who participate in different school clubs. Each student is a member of only one club.



(a) Complete the Pareto chart on the diagram above by adding the final column to represent membership of the bowling club. bowling = 50 − 22 − 18 − 8 = 2 [1]

(b) Which club has 16% of the students? [1]

16% of 50 = 8 students

Ceramics ✓

QUESTION FOURTEEN (2 marks) **Marks**

Calculate the limiting sum of the geometric series $100 + 20 + 4 + \cdots$. [2]

$a = 100$, $r = \frac{1}{5}$ ✓ for correct r

$S_\infty = \dfrac{100}{1 - \frac{1}{5}}$

$= 125$ ✓ showing use of formula

QUESTION FIFTEEN (3 marks) **Marks**

Consider the function $f(x) = \ln(x-3)$.

(a) Find $f'(x)$. [1]

$f'(x) = \dfrac{1}{x-3}$ ✓

(b) Hence, find the equation of the tangent to the graph of $y = f(x)$ where $x = 4$. [2]

$f(4) = \ln(4-3)$

$= 0$

$f'(4) = \dfrac{1}{4-3}$

$= 1$ ✓

$m_{tan} = 1$, passes through $(4, 0)$

$y - 0 = 1(x-4)$

$y = x - 4$ ✓

QUESTION SIXTEEN (2 marks) **Marks**

The probability distribution of a discrete random variable X is given in the table below. [2]

x	4	9	14	a
$P(X = x)$	0·2	0·3	0·4	0·1

Find the value of a given that $E(X) = 10{\cdot}6$.

$4 \times 0.2 + 9 \times 0.3 + 14 \times 0.4 + 0.1a = 10.6$ ✓

$9.1 + 0.1a = 10.6$

$0.1a = 1.5$

$a = 15$ ✓ note: this is a discrete probability distribution, so you should expect a to be an integer

QUESTION SEVENTEEN (2 marks) **Marks**

Solve $3^{3x-1} = 27$. [2]

$3^{3x-1} = 3^3$

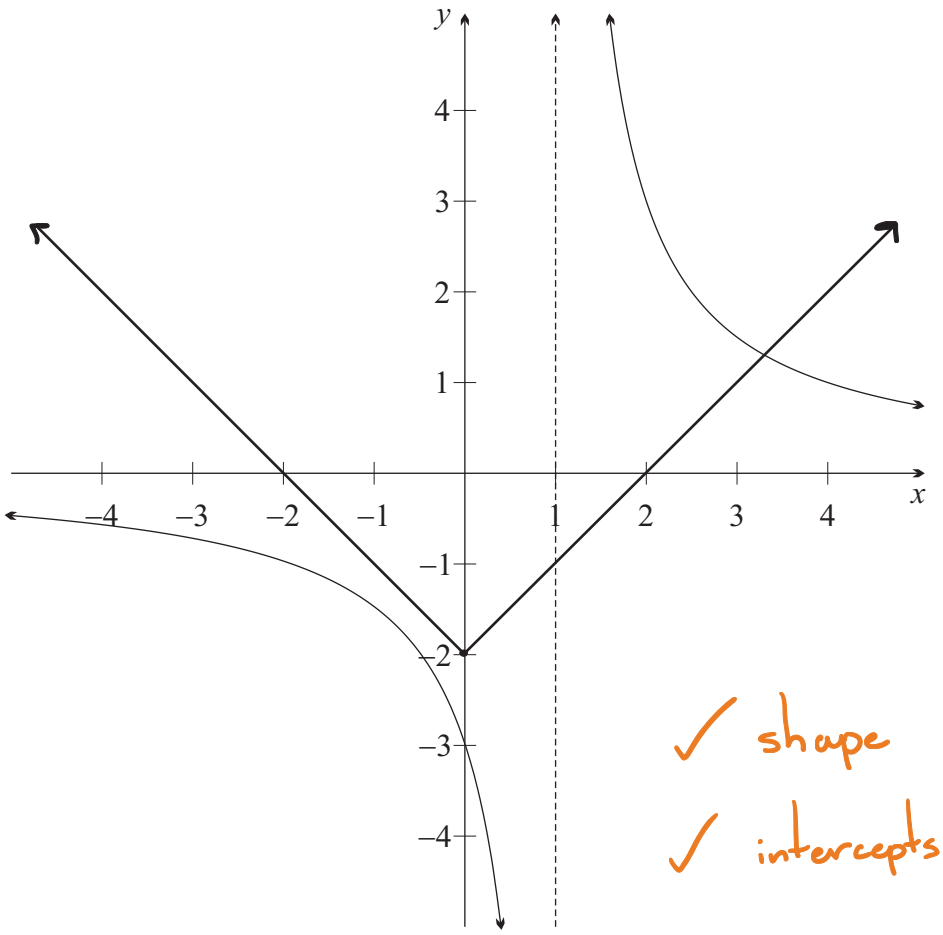
$3x - 1 = 3$ ✓

$3x = 4$

$x = \frac{4}{3}$ ✓

QUESTION EIGHTEEN (3 marks) **Marks**

The function $y = \dfrac{3}{x-1}$ has been graphed on the coordinate plane below.



✓ shape

✓ intercepts

(a) Sketch the graph of $y = |x| - 2$ on the coordinate plane above, clearly showing any intercepts with the axes. [2]

(b) Hence, determine the number of solutions of the equation $|x| - \dfrac{3}{x-1} = 2$. [1]

1 point of intersection, so 1 solution ✓

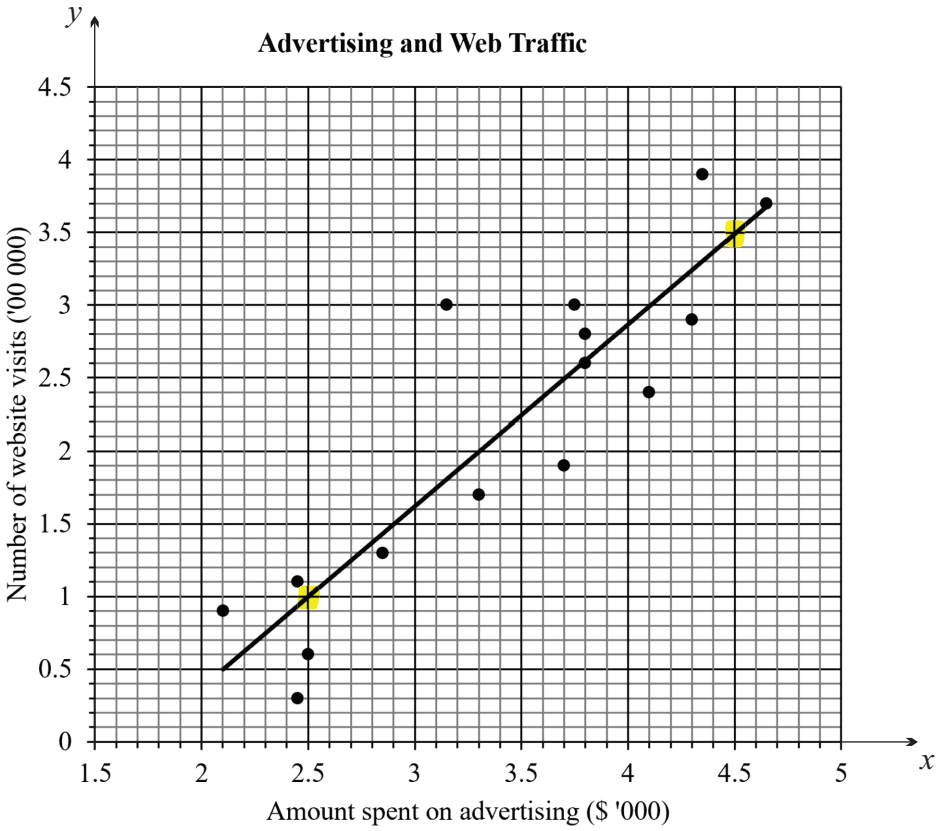
QUESTION NINETEEN (5 marks) **Marks**

A company wanted to investigate the strength of the relationship between the amount of money spent on advertising each week and the number of visitors the company's website receives each week. Data was collected over a 15 week period.

(a) Pearson's correlation coefficient for the data set is $r = 0{\cdot}9$. Describe the strength and direction of this linear correlation. [1]

strong, positive ✓

(b) The graph below shows a scatterplot and the regression line for the data, where x is the amount of money spent on advertising in thousands of dollars and y is the website traffic in hundreds of thousands of visits. [2]



Calculate the equation of the regression line.

$m = \dfrac{3.5-1}{4.5-2.5}$

$m = 1.25$ ✓ using two points to find gradient

passes through $(2.5, 1)$

$y - 1 = 1.25(x - 2.5)$

$y = 1.25x - 3.125 + 1$

$y = 1.25x - 2.125$ ✓ equation

note: it is possible to use the points to find the equation of the regression line, but this method was time consuming and error prone. must have correct equation to earn full marks for this method.

(c) Use the equation of your regression line from part (b) to predict the number of visitors the website will receive for a week in which \$3000 was spent on advertising. [1]

when $x = 3$,

$y = 1.25 \times 3 - 2.125$

$= 1.625$

$\therefore$ 162 500 visitors ✓

(d) Explain why the model is not useful for predicting the number of visitors for weeks in which \$1500 was spent on advertising. [1]

it is extrapolation ✓

or when $x = 1.5$, $y = 1.25 \times 1.5 - 2.125 = -0.25$, which doesn't make sense as it is impossible to have a negative number of visitors

QUESTION TWENTY (2 marks) **Marks**

Find these indefinite integrals.

(a) $\int (6x^2 + 5)\, dx$ [1]

$= \frac{6}{3} x^3 + 5x + c$

$= 2x^3 + 5x + c$ ✓

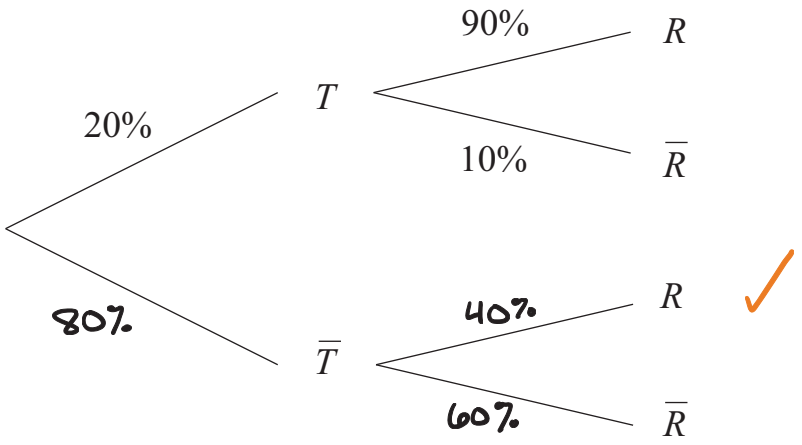
(b) $\int e^{4x}\, dx$ [1]

$= \frac{1}{4} e^{4x} + c$ ✓

QUESTION TWENTY-ONE (3 marks) **Marks**

Briana is on a hiking trip in the outback. She is within 50 km of a town 20% of the time. When she is within 50 km of a town, she has mobile reception 90% of the time. Otherwise, she has mobile reception 40% of the time.

(a) Let T be the event that Briana is within 50 km of a town and R be the event that she has mobile reception. Complete the probability tree diagram below by writing the missing probabilities on the branches. 1



(b) Calculate the probability that she has mobile reception at any given time. 2

$P(\text{has reception}) = P(TR) + P(\overline{T}R)$

$= 0.2 \times 0.9 + 0.8 \times 0.4$

$= 0.5$

$= 50\%$

QUESTION TWENTY-TWO (2 marks) **Marks**

Expand and simplify $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$. 2

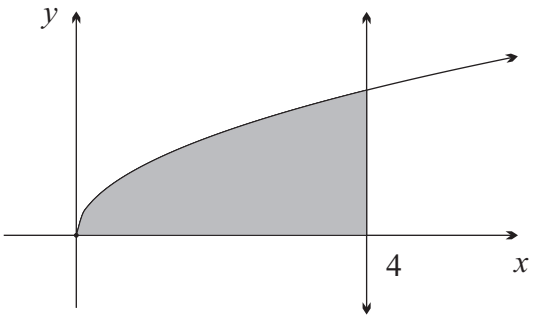
$(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$

$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 2\sin\theta\cos\theta$

$= 1$

QUESTION TWENTY-THREE (2 marks) **Marks**

The diagram below shows the graph of the function $y = \sqrt{x}$. The shaded region is bounded by the curve, the x-axis, and the line $x = 4$. Calculate the area of the shaded region. [2]



$A = \int_0^4 \sqrt{x}\, dx$

$= \int_0^4 x^{\frac{1}{2}}\, dx$

$= \left[\frac{2}{3} x^{\frac{3}{2}}\right]_0^4$ ✓

$= \frac{2}{3}\left(4^{\frac{3}{2}} - 0\right)$

$= \frac{2}{3} \times 8$

$= \frac{16}{3}\ u^2$ ✓

QUESTION TWENTY-FOUR (3 marks) **Marks**

The sum of the first n terms of an arithmetic series is S_n. For a particular arithmetic series, $S_8 = 196$ and $S_9 = 279$.

(a) Calculate the value of the 9^{th} term. [1]

$T_9 = S_9 - S_8$

$= 279 - 196$

$= 83$ ✓

(b) Hence, find the values of the first term a and the common difference d. [2]

$T_9 = 83$ $\qquad$ $S_9 = 279$

$83 = a + (9-1)d$ $\qquad$ $279 = \frac{9}{2}(a + 83)$

$83 = a + 8d$ $\qquad$ $62 = a + 83$

$a = -21$ ✓

sub $a = -21$

$83 = -21 + 8d$

$8d = 104$

$d = 13$ ✓

QUESTION TWENTY-FIVE (1 mark) **Marks**

If $\int_{-10}^{4} f(x)\,dx = 3$ and $\int_{4}^{6} f(x)\,dx = 5$, find the value of $\int_{-10}^{6} 2f(x)\,dx$. [1]

$\int_{-10}^{6} 2f(x)\,dx = 2\left[\int_{-10}^{4} f(x)\,dx + \int_{4}^{6} f(x)\,dx\right]$

$= 2[3 + 5] = 16$ ✓

QUESTION TWENTY-SIX (3 marks) **Marks**

The Richter scale is used to measure the magnitude of earthquakes. The magnitude R of an earthquake is proportional to the logarithm of the ground displacement d caused by the earthquake in millimetres. This relationship can be described by the equation:

$$R = \log_{10}(1000d)$$

(a) In July 2022, an earthquake in the Easter Island region caused a ground displacement of 6310 mm. What was its magnitude on the Richter scale? Give your answer correct to one decimal place. [1]

$R = \log_{10}(1000 \times 6310)$

$= 6.80002\ldots$

$\doteq 6.8$ ✓

(b) How many times larger is the ground displacement of an earthquake with a magnitude of 3·2 compared to an earthquake with a magnitude of 1·6? Give your answer correct to the nearest whole number. [2]

$3.2 = \log_{10}(1000d_1)$

$d_1 = \dfrac{10^{3.2}}{1000}$

$1.6 = \log_{10}(1000d_2)$

$d_2 = \dfrac{10^{1.6}}{1000}$ ✓

$\dfrac{d_1}{d_2} = \dfrac{\frac{10^{3.2}}{1000}}{\frac{10^{1.6}}{1000}}$

$= \dfrac{10^{3.2}}{10^{1.6}}$

$\doteq 40$ times ✓

QUESTION TWENTY-SEVEN (1 mark) **Marks**

The graph of the function $y = \dfrac{x^2 + 3x}{x^2 + 3}$ has a horizontal asymptote. Find the equation of this asymptote, clearly showing your working. [1]

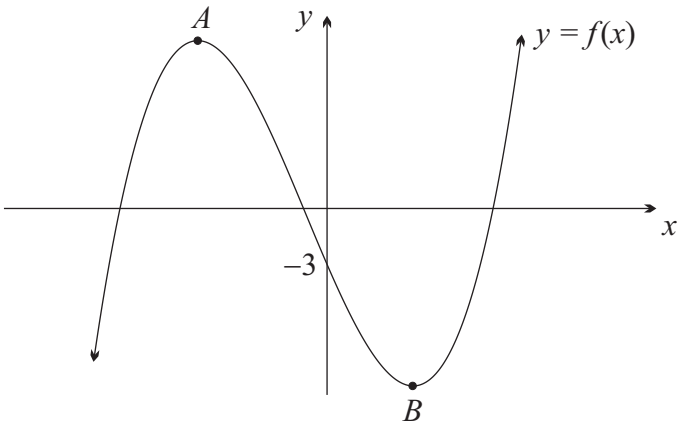
$y = \dfrac{1 + \frac{3}{x}}{1 + \frac{3}{x^2}}$

as $x \to \pm\infty$, $y \to 1$

$\therefore$ horizontal asymptote has equation $y = 1$ ✓ with supporting working

QUESTION TWENTY-EIGHT (6 marks) **Marks**

The diagram below shows the graph of a function $f(x)$ with a y-intercept of -3, a maximum turning point at A, and a minimum turning point at B. The derivative of the function is $f'(x) = 3x^2 + 2x - 8$.



(a) Find the equation of $f(x)$. [2]

$f'(x) = 3x^2 + 2x - 8$

$f(x) = x^3 + x^2 - 8x + c$ ✓

when $x = 0$, $y = -3$

$-3 = 0 + 0 - 0 + c$

$c = -3$

$f(x) = x^3 + x^2 - 8x - 3$ ✓

(b) Find the coordinates of A. 3

stationary points when $f'(x) = 0$

$0 = 3x^2 + 2x - 8$

$0 = (3x - 4)(x + 2)$

$x = \frac{4}{3}, -2$ ✓

$f''(x) = 6x + 2$

$f''(-2) = 6 \times (-2) + 2 = -10$, so max at $x = -2$

as A is to the left of the y-axis, x-coordinate of A is -2 ✓

$f(-2) = (-2)^3 + (-2)^2 - 8(-2) - 3$

$= 9$

$A(-2, 9)$ ✓

(c) The graph of $y = g(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis, shifting 5 units up, and then dilating horizontally by a factor of $\frac{1}{2}$. 2
Find the coordinates of the point A', the image of the point A, after these successive transformations have been applied.

$A'\left(-2 \times \frac{1}{2}, -9 + 5\right)$

$A'(-1, -4)$ ✓ ✓

QUESTION TWENTY-NINE (2 marks) **Marks**

Consider the exact values $\sin\frac{\pi}{8} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$ and $\cos\frac{\pi}{8} = \dfrac{(\sqrt{2}-1)\sqrt{2+\sqrt{2}}}{2}$. 2

Show that $-\sin\frac{\pi}{8} - \cos\frac{\pi}{8} = -\dfrac{\sqrt{4+2\sqrt{2}}}{2}$.

LHS $= -\sin\frac{\pi}{8} - \cos\frac{\pi}{8}$

$= -\dfrac{\sqrt{2+\sqrt{2}}}{2} - \dfrac{(\sqrt{2}-1)\sqrt{2+\sqrt{2}}}{2}$

$= \dfrac{-\sqrt{2+\sqrt{2}} - \sqrt{2}\sqrt{2+\sqrt{2}} + \sqrt{2+\sqrt{2}}}{2}$ ✓

$= -\dfrac{\sqrt{2}\sqrt{2+\sqrt{2}}}{2}$

$= -\dfrac{\sqrt{2(2+\sqrt{2})}}{2}$

$= -\dfrac{\sqrt{4+2\sqrt{2}}}{2}$ ✓

QUESTION THIRTY (7 marks) **Marks**

Consider the function $f(x) = x^3 - 4x^2 + 4x$.

(a) Find the coordinates of the stationary points and determine their nature. 4

$f'(x) = 3x^2 - 8x + 4$

stationary points occur when $f'(x) = 0$

$0 = 3x^2 - 8x + 4$

$0 = (3x - 2)(x - 2)$

$x = \frac{2}{3}, 2$ ✓ x - values

$f(\frac{2}{3}) = (\frac{2}{3})^3 - 4(\frac{2}{3})^2 + 4(\frac{2}{3})$ $f(2) = 2^3 - 4\times 2^2 + 4\times 2$

$= \frac{32}{27}$ $= 0$ ✓ y-values

$f''(x) = 6x - 8$

$f''(\frac{2}{3}) = 6(\frac{2}{3}) - 8$ $f''(2) = 6(2) - 8$

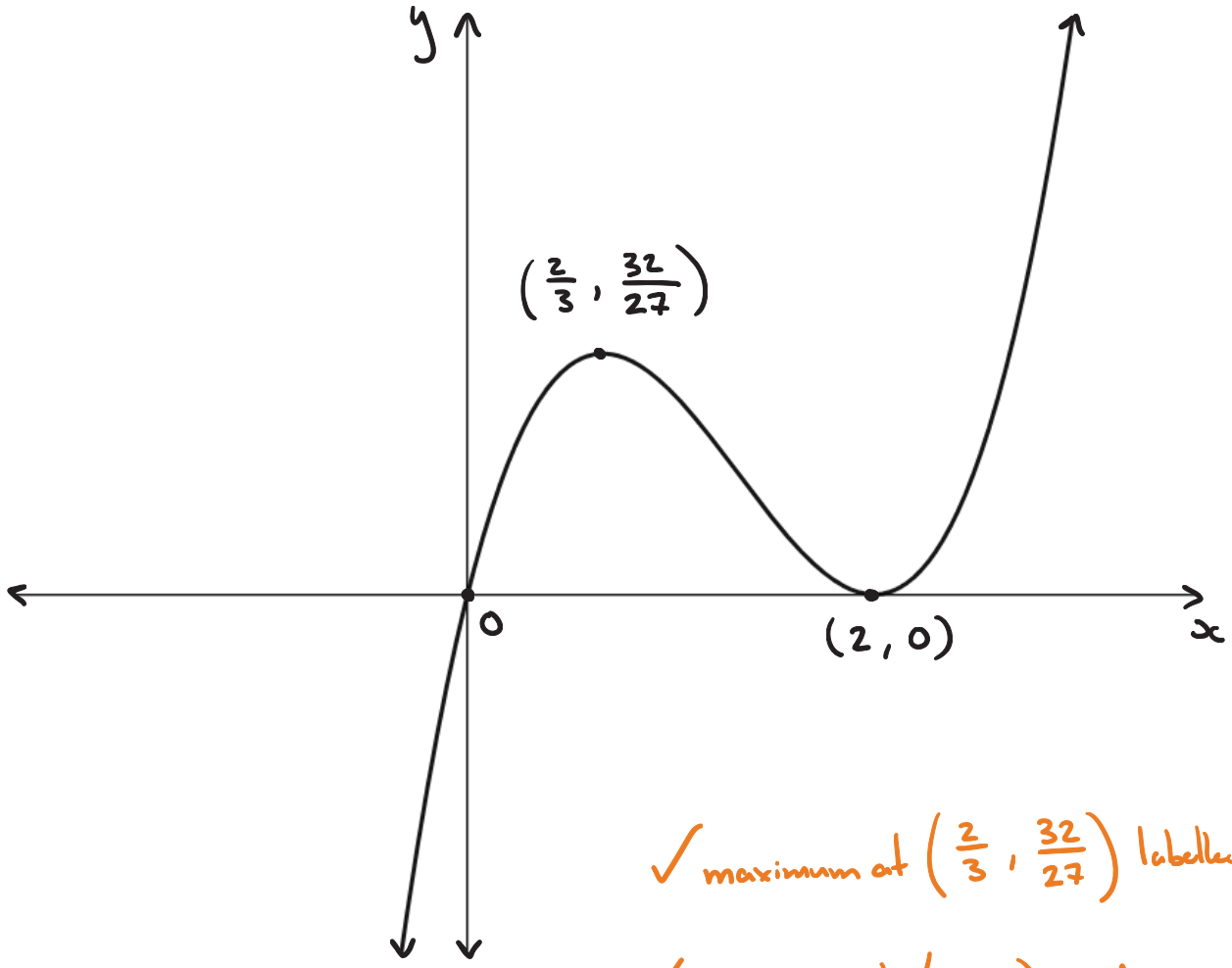
$= -4$ $= 4$ ✓ substantial start to determine nature

$\therefore$ concave down $\therefore$ concave up

$\therefore$ max turning point $\therefore$ min turning point

at $(\frac{2}{3}, \frac{32}{27})$ at $(2, 0)$ ✓ nature of stationary points

(b) Sketch the graph of $y = f(x)$, clearly showing any intercepts and stationary points. [2]



✓ maximum at $\left(\frac{2}{3}, \frac{32}{27}\right)$ labelled clearly

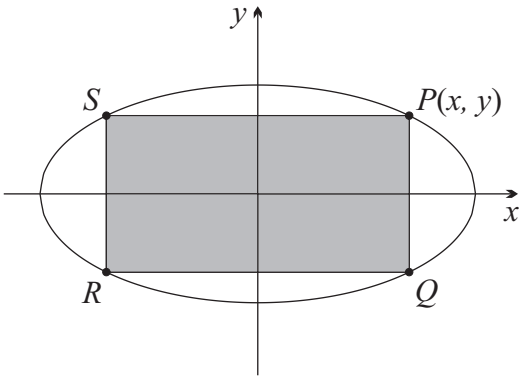
✓ minimum at (2, 0) and passes through (0, 0)

(c) Hence, determine the values of k for which the equation $x^3 - 4x^2 + 4x + k = 0$ has exactly two solutions. [1]

$k = 0, -\frac{32}{27}$ ✓

QUESTION THIRTY-ONE (7 marks) **Marks**

Rectangle $PQRS$ has been inscribed inside an ellipse with equation $x^2 + 4y^2 = 36$, as shown in the diagram below. The point P has coordinates (x, y).



(a) Show that the area of the rectangle is $A = 2x\sqrt{36 - x^2}$. 2

$x^2 + 4y^2 = 36$

$y^2 = \frac{36 - x^2}{4}$

$y = \frac{\sqrt{36 - x^2}}{2}$ ✓

reject -ve as P is in 1st quadrant

$A = PS \times PQ$

$= 2x \times 2y$

$= 2x \times \frac{2\sqrt{36 - x^2}}{2}$

$= 2x\sqrt{36 - x^2}$ ✓

(b) Show that $\frac{dA}{dx} = \frac{72 - 4x^2}{\sqrt{36 - x^2}}$. 2

let $u = 2x$ $\quad v = (36 - x^2)^{\frac{1}{2}}$

$u' = 2$ $\quad v' = \frac{1}{2}(36 - x^2)^{-\frac{1}{2}} \times (-2x)$

$= -\frac{x}{\sqrt{36 - x^2}}$

$\frac{dA}{dx} = vu' + uv'$

$= 2\sqrt{36 - x^2} - \frac{2x^2}{\sqrt{36 - x^2}}$ ✓

$= \frac{2(36 - x^2) - 2x^2}{\sqrt{36 - x^2}} = \frac{72 - 4x^2}{\sqrt{36 - x^2}}$ ✓

(c) Hence, find the maximum possible area of the rectangle. 3

max area when $\frac{dA}{dx} = 0$

$$0 = \frac{72 - 4x^2}{\sqrt{36 - x^2}}$$

$$0 = 72 - 4x^2$$

$$4x^2 = 72$$

$$x^2 = 18$$

$$x = \pm 3\sqrt{2}$$

✓ find x-values of stationary points

reject $-3\sqrt{2}$, as x is in 1st quadrant

x	1	$3\sqrt{2}$	5
$\frac{dA}{dx}$	$\frac{68}{\sqrt{35}}$	0	$-\frac{28}{\sqrt{11}}$
	/	—	\

∴ maximum area when $x = 3\sqrt{2}$ ✓ show that $x = 3\sqrt{2}$ gives max area

max area is $A = 2 \times 3\sqrt{2} \times \sqrt{36 - (3\sqrt{2})^2}$

$= 6\sqrt{2} \times \sqrt{36 - 18}$

$= 6\sqrt{2} \times 3\sqrt{2}$

$= 36 u^2$ ✓ area

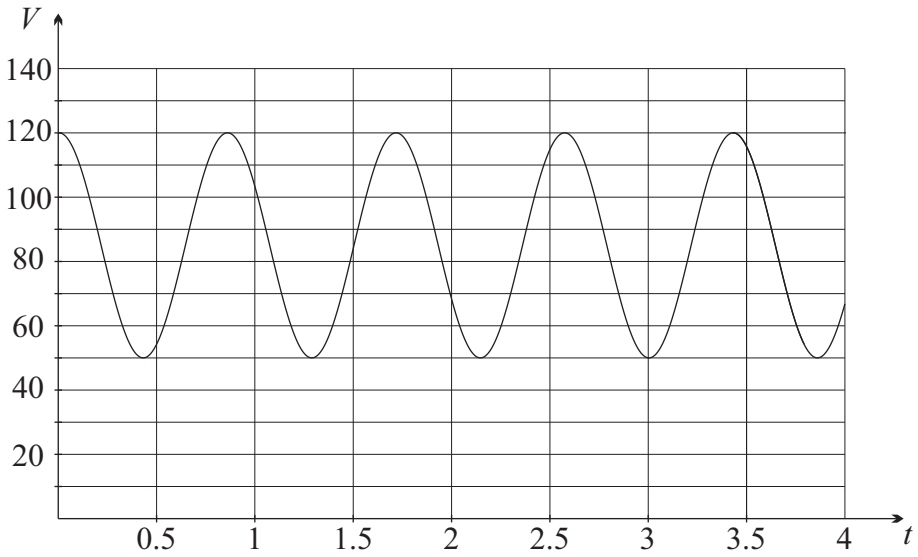
note: possible to show that $x = 3\sqrt{2}$ gives maximum area by using $\frac{d^2A}{dx^2}$, but this method required substantially more algebra and was much more error prone

QUESTION THIRTY-TWO (8 marks) **Marks**

The human heart pumps blood through the body by repeatedly contracting, which pushes blood out of the heart, and then relaxing, which lets new blood flow into the heart. The average human heart contains 50 mL of blood when contracted and 120 mL when relaxed. The total volume V mL of blood in the heart at time t seconds can be modelled by the function

$$V = 35\cos\left(\frac{7\pi}{3}t\right) + c$$

where c is a constant. The diagram below shows the graph of $V(t)$ for $0 \leq t \leq 4$.



(a) Write down the value of c. 1

$c = 85$ ✓

(b) Calculate the period of V. 1

$b = \frac{2\pi}{T}$, where b is coefficient of t and period is T

$\frac{7\pi}{3} = \frac{2\pi}{T}$

$T \times \frac{7\pi}{3} = 2\pi$

$T \times 7\pi = 6\pi$

$T = \frac{6}{7}$ ✓ $\therefore$ period is $\frac{6}{7}$ s

(c) Calculate the rate of change of volume when $t = \frac{1}{2}$. Give your answer correct to the nearest millilitre per second. [2]

$\frac{dV}{dt} = -35 \times \frac{7\pi}{3} \sin\left(\frac{7\pi t}{3}\right)$

when $t = \frac{1}{2}$,

$\frac{dV}{dt} = -\frac{245\pi}{3} \sin\left(\frac{7\pi}{6}\right)$ ✓ correct derivative and substitution of $t = \frac{1}{2}$

$= -\frac{245\pi}{3} \times \left(-\frac{1}{2}\right)$

$= \frac{245\pi}{6}$

$\fallingdotseq 128$ mL/s ✓

(d) What is the total volume of blood which is pumped out of the heart between $t = \frac{2}{7}$ and $t = \frac{3}{7}$? Give your answer correct to the nearest millilitre. [2]

volume when $t = \frac{2}{7}$

$V = 35\cos\left(\frac{7\pi}{3} \times \frac{2}{7}\right) + 85$

$= 35\cos\left(\frac{2\pi}{3}\right) + 85$

$= 35\left(-\frac{1}{2}\right) + 85$

$= 67.5$ ✓

volume when $t = \frac{3}{7}$ occurs halfway through a period, so volume will be at its minimum

$V = 50$

volume pumped out $= 67.5 - 50$

$= 17.5$

$\fallingdotseq 18$ mL ✓ ignore rounding error

(e) By considering the volume of blood in the heart between $t = 0$ and $t = \frac{3}{7}$, or otherwise, determine what fraction of the time the heart contains at least 102·5 mL of blood. [2]

when $V = 102.5$,

$102.5 = 35\cos\left(\frac{7\pi t}{3}\right) + 85$

$17.5 = 35\cos\left(\frac{7\pi t}{3}\right)$

$\cos\left(\frac{7\pi t}{3}\right) = \frac{1}{2}$

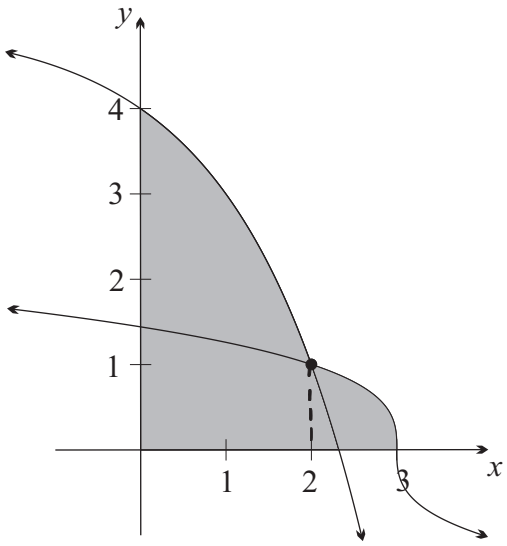
$\frac{7\pi t}{3} = \frac{\pi}{3}$ is the first time this occurs

$t = \frac{1}{7}$ ✓

fraction of time is $\frac{\frac{1}{7}}{\frac{3}{7}} = \frac{1}{3}$ ✓

QUESTION THIRTY-THREE (4 marks) **Marks**

The diagram below shows the graphs of $y = 5 - 2^x$ and $y = \sqrt[3]{3-x}$. The graphs intersect at the point $(2, 1)$. Calculate the exact area of the shaded region. [4]



$A = \int_0^2 (5 - 2^x)\,dx + \int_2^3 (3-x)^{\frac{1}{3}}\,dx$ ✓ correct expression for area

$= \left[5x - \frac{1}{\ln 2} \times 2^x\right]_0^2 + \left[-\frac{3}{4}(3-x)^{\frac{4}{3}}\right]_2^3$ ✓ for each primitive

$= 5(2) - \frac{1}{\ln 2} \times 2^2 - \left(5(0) - \frac{1}{\ln 2} \times 2^0\right) - \left[\frac{3}{4}(3-3)^{\frac{4}{3}} - \frac{3}{4}(3-2)^{\frac{4}{3}}\right]$

$= 10 - \frac{4}{\ln 2} - 0 + \frac{1}{\ln 2} - \left(0 - \frac{3}{4} \times 1\right)$

$= 10\frac{3}{4} - \frac{3}{\ln 2}\ u^2$ ✓ simplified exact value

QUESTION THIRTY-FOUR (3 marks) **Marks**

Show that the function $y = kxe^{kx}$ has exactly one stationary point and its y-value is independent of k, where k is a constant. [3]

let $u = kx$ $\quad v = e^{kx}$

$u' = k$ $\quad v' = ke^{kx}$

$y' = vu' + uv'$

$= ke^{kx} + k^2xe^{kx}$

$= ke^{kx}(1 + kx)$ ✓ derivative

stationary points occur when $y' = 0$

$0 = ke^{kx}(1 + kx)$

$ke^{kx} = 0$, $\quad 1 + kx = 0$

no solution , $\quad kx = -1$

$x = -\frac{1}{k}$ is the only stationary point ✓ x-value

when $x = -\frac{1}{k}$, $\quad y = k\left(-\frac{1}{k}\right)e^{k\left(-\frac{1}{k}\right)}$

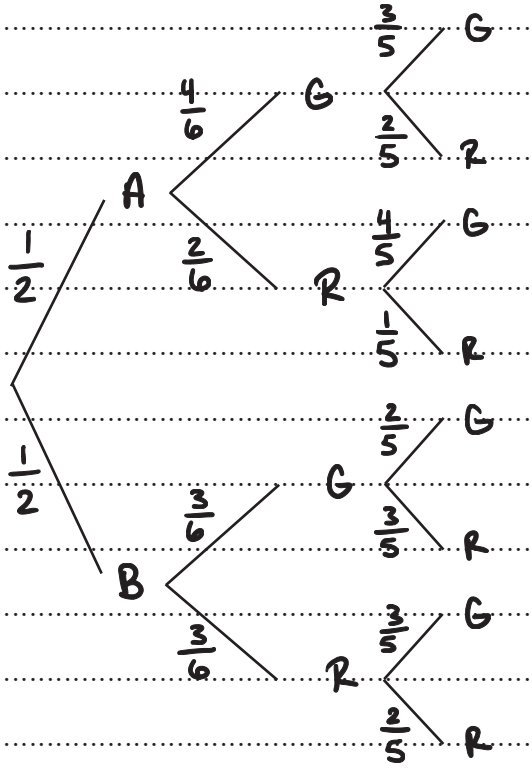
$= -e^{-1}$, which is independent of k

✓ y-value

QUESTION THIRTY-FIVE (3 marks) **Marks**

Arthur's bag contains 4 green marbles and 2 red marbles. Beatrice's bag contains 3 green marbles and 3 red marbles. One of their bags is randomly selected, and then two marbles are removed in succession without replacement. 3

What is the probability that Arthur's bag was selected, given that the two marbles were the same colour?



✓ tree diagram or other substantial start

$P(\text{Arthur's bag} \mid \text{same colour})$

$= \dfrac{P(\text{Arthur's bag and same colour})}{P(\text{same colour})}$

$= \dfrac{P(ARR \text{ or } AGG)}{P(ARR \text{ or } AGG \text{ or } BRR \text{ or } BGG)}$

$P(ARR \text{ or } AGG)$

$= \frac{1}{2} \times \frac{2}{6} \times \frac{1}{5} + \frac{1}{2} \times \frac{4}{6} \times \frac{3}{5}$

$= \frac{7}{30}$

$P(\text{same colour})$

$= P(ARR \text{ or } AGG) + P(BRR \text{ or } BGG)$

$= \frac{7}{30} + \frac{1}{2} \times \frac{3}{6} \times \frac{2}{5} + \frac{1}{2} \times \frac{3}{6} \times \frac{2}{5}$

$= \frac{13}{30}$

✓ a correct, necessary probability calculation

$P(\text{Arthur's bag} \mid \text{same colour}) = \dfrac{\frac{7}{30}}{\frac{13}{30}}$

$= \frac{7}{13}$

✓ conditional probability

QUESTION THIRTY-SIX (10 marks) **Marks**

A table of present value interest factors for an annuity of \$1 is shown.

Table of present value interest factors

	Rate per time period			
Number of periods	0·05%	0·1%	0·15%	0·2%
120	116·44	113·03	109·74	106·59
180	172·10	164·65	157·64	151·04
240	226·11	213·28	201·42	190·46
300	278·52	259·07	241·44	225·43
360	329·38	302·20	278·01	256·45

On 1st April 2020, Ferdinand takes out a 25 year loan with reducible interest charged at a rate of 1·2% per annum, compounded monthly.
The loan is to be repaid in equal monthly repayments of \$2370 at the end of each month.

(a) Use the present value table to show that Ferdinand borrowed \$614 000, correct to the nearest thousand dollars. 1

amount borrowed = 2370×259.07
$= 613995.9$
$\doteqdot \$614\,000$

✓ showing correct use of present value table

(b) Hence, calculate the total amount of interest Ferdinand will pay over the 25 year term of the loan. 2

total paid $= 2370 \times 300$
$= 711\,000$ ✓

interest $= 711000 - 614000$
$= \$97\,000$ ✓

(c) Let A_n be the amount left to repay at the end of n months. [1]

Show that $A_2 = 614\,000 \times 1{\cdot}001^2 - 2370\,(1 + 1{\cdot}001)$.

$A_1 = 614000 \times 1.001 - 2370$

$A_2 = A_1 \times 1.001 - 2370$

$= (614000 \times 1.001 - 2370) \times 1.001 - 2370$

$= 614000 \times 1.001^2 - 2370 \times 1.001 - 2370$

$= 614000 \times 1.001^2 - 2370(1 + 1.001)$ ✓ with supporting working

(d) Show that $A_n = 614\,000 \times 1{\cdot}001^n - 2\,370\,000\,(1{\cdot}001^n - 1)$. [2]

$A_3 = 614000 \times 1.001^3 - 2370(1 + 1.001 + 1.001^2)$

⋮

✓ for A_3 or A_n

$A_n = 614000 \times 1.001^n - 2370(1 + 1.001 + 1.001^2 + \ldots + 1.001^{n-1})$

$= 614000 \times 1.001^n - 2370 \times \dfrac{1\,(1.001^n - 1)}{1.001 - 1}$

$= 614000 \times 1.001^n - \dfrac{2370(1.001^n - 1)}{0.001}$

$= 614000 \times 1.001^n - 2370000(1.001^n - 1)$ ✓ with correct supporting working

(e) Hence, calculate the amount owing at the end of March 2022. Give your answer correct to the nearest dollar. [1]

$A_{24} = 614000 \times 1.001^{24} - 2370000(1.001^{24} - 1)$

$= 571367.7711\ldots$

$\doteqdot \$571367$ ✓

(f) On 1st April 2022, rising inflation and the outbreak of a conflict in Europe have economic impacts which cause the bank to increase the interest rate on home loans to 3·6% per annum. Calculate the amount of Ferdinand's new monthly repayment, correct to the nearest dollar, if he is still to repay the loan within the 25 year term. [3]

let B_n be the amount owing n months after 1st April 2022

$r = 3.6\%$ p.a. $= 0.3\%$ per month, M is new monthly repayment

$$B_n = 571367 \times 1.003^n - \frac{M(1.003^n - 1)}{1.003 - 1}$$ ✓ new formula

when $n = 276$, $B_n = 0$

$$0 = 571367 \times 1.003^{276} - \frac{M(1.003^{276} - 1)}{0.003}$$ ✓ $n, B_{276} = 0$

$$\frac{M(1.003^{276} - 1)}{0.003} = 571367 \times 1.003^{276}$$

$$M = \frac{571367 \times 1.003^{276} \times 0.003}{1.003^{276} - 1}$$

$= 3047.09656\ldots$

$\doteqdot \$3047.10$ ✓ monthly repayment

END OF PAPER